Free quasi-symmetric functions, product actions and quantum field theory of partitions.

Gérard Henry Edmond Duchamp, Jean-Gabriel Luque, Karol A. Penson, Christophe Tollu

Free quasi-symmetric functions, product actions and quantum field theory of partitions.

Gérard H. E. Duchamp †, **Jean-Gabriel Luque** ‡, **K A Penson** ⋆ **Christophe Tollu**†,
(†) LIPN, UMR CNRS 7030
Institut Galilée - Université Paris-Nord
99, avenue Jean-Baptiste Clément
93430 Villetaneuse, France
(‡) IGM, Laboratoire d'informatique UMR 8049 IGM-LabInfo
77454 Marne-la-Vallée Cedex 2, France.
⋆ LPTL, Laboratoire de Physique Théorique des Liquides, CNRS UMR 7600
Université Pierre et Marie Curie,
Tour 16, $5^{ième}$ étage, 4, place Jussieu, F 75252 Paris Cedex 05, France
{ghed@lipn.univ-paris13.fr, luque@univ-mlv.fr,
penson@lptl.jussieu.fr, ct@lipn.univ-paris13.fr}

December 13, 2004

Abstract: We examine two associative products over the ring of symmetric functions related to the intransitive and Cartesian products of permutation groups. As an application, we give an enumeration of some Feynman type diagrams arising in Bender's QFT of partitions. We end by exploring possibilities to construct noncommutative analogues.

Résumé: Nous étudions deux lois produits associatives sur les fonctions symétriques correspondant aux produits intransitif et cartsien des groupes de permutations. Nous donnons comme application l'énumération de certains diagrammes de Feynman apparaissant dans la QFT des partitions de Bender. Enfin, nous donnons quelques pistes possibles pour construire des analogues non-commutatifs.

1 Introduction

In a relatively recent paper, Bender, Brody and Meister introduce a special Field Theory described by

$$G(z) = \left(e^{(\sum_{n\geq 1} L_n \frac{z^n}{n!}\frac{\partial}{\partial x})}\right)\left(e^{(\sum_{m\geq 1} V_m \frac{x^m}{m!})}\right)\Big|_{x=0} \qquad (1)$$

in order to prove that any sequence of numbers $\{a_n\}$ can be generated by a suitable set of rules applied to some type of Feynman diagrams [1, 2]. These diagrams actually are bipartite finite graphs with no isolated vertex, and edges weighted with integers.

Expanding one factor of (1), we can observe surprising links between: the normal ordering problem (for bosons), the parametric Stieltjes moment problem and the convolution of kernels, substitution matrices (such as generalised Stirling matrices) and one-parameter groups of analytic substitutions [7, 8, 13].

The aim of this paper is to make explicit the multifaceted connections between noncommutative symmetric functions (here **MQSym**, **FQSym** [5]) and the Feynman diagrams arising in the expansion of formula (1) used in combinatorial physics [13].

The structure of the contribution is the following. In Section 2, we define two associative products in $\mathfrak{S} = \bigsqcup \mathfrak{S}_n$ related to the Intransitive and Cartesian products of permutation groups. These products induce a structure of 2-associative algebra over the symmetric functions. The properties of this algebra are investigated in Section 3. At the end of this section, we give, as an application, an inductive formula for computing generating series of Bender's Feynman diagrams. Noncommutative analogues are proposed in Section 4.

2 Actions of a direct product of permutation groups

2.1 Direct product actions

The actions of the direct product of two permutation groups (in particular, the structure of the cycles) give rise to interesting properties related to the enumeration of unlabelled objets [12]. We open this section with the definition of two actions (namely, Intransitive and Cartesian). For greater detail about these constructions (or for constructions involving the wreath product) the reader can refer to [3].

Consider two pairs (G_1, X_1) and (G_2, X_2), where each G_i is a permutation group acting on X_i. The *intransitive action* of $G_1 \times G_2$ on $X_1 \sqcup X_2$ (here $\sqcup$ means disjoint union) is defined by the rule

$$(\sigma_1, \sigma_2)x = \begin{cases} \sigma_1 x & \text{if } x \in X_1 \\ \sigma_2 x & \text{if } x \in X_2 \end{cases}. \tag{2}$$

This action will be denoted by $(G_1, X_1) \nrightarrow (G_2, X_2) := (G_1 \times G_2, X_1 \sqcup X_2)$.

The *Cartesian action* of $G_1 \times G_2$ on $X_1 \times X_2$ is defined by

$$(\sigma_1, \sigma_2)(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2). \tag{3}$$

This action will be denoted by $(G_1, X_1) \times\!\!\!\!\!\times (G_2, X_2) := (G_1 \times G_2, X_1 \times X_2)$. Note that neither of the two laws just defined is commutative. A natural question to ask is whether such a structure enjoys some algebraic properties. For example, is the $\times\!\!\!\!\!\times$ law distributive over $\nrightarrow$?

In other words, what is the meaning of

$$(G_1, X_1) \times\!\!\!\!\!\times ((G_2, X_2) \nrightarrow (G_3, X_3)) = (G_1 \times G_2 \times G_3, X_1 \times (X_2 \sqcup X_3))$$

and

$$((G_1, X_1) \times\!\!\!\!\!\times (G_2, X_2)) \nrightarrow ((G_1, X_1) \times\!\!\!\!\!\times (G_3, X_3)) = (G_1 \times G_2 \times G_1 \times G_3, (X_1 \times X_2) \sqcup (X_1 \times X_3)).$$

The groups $G_1 \times G_2 \times G_1 \times G_3$ and $G_1 \times G_2 \times G_3$ are not isomorphic, so distributivity does not hold, although the set-theoretical Cartesian product is distributive over disjoint union. However an examination of the structure of the cycles (see [3] for the general construction or section 2.2 for a particular case) shows that the cycles are the same. More precisely, a cycle can appear with different multiplicities according to which group is acting, but if we focus on the set of the cycles, the two structures are similar.

Now, let us give a construction which takes such a phenomenon into account.

2.2 Explicit realization

We will denote by $\circ_N$ the natural action of $\mathfrak{S}_n$ on $\{0, \ldots, n-1\}$. Let $\mathfrak{S}_n$ and $\mathfrak{S}_m$ be two symmetric groups, we note by $\circ_I$ the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \cdots, n+m-1\}$ and by $\circ_C$ the *Cartesian action* of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \ldots, nm-1\}$. More precisely,

$$(\sigma_1, \sigma_2) \circ_I i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \le i \le n-1 \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \le i \le n+m-1 \end{cases}. \tag{4}$$

for $0 \le i \le n+m-1$, and

$$(\sigma_1, \sigma_2) \circ_C (j + nk) = (\sigma_1 \circ_N j) + n(\sigma_2 \circ_N k) \tag{5}$$

for $0 \le j \le n-1$ and $0 \le k \le m-1$.

The *intransitive product* is the map $\nrightarrow : \mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ defined by

$$\sigma_1 \nrightarrow \sigma_2 = \sigma_1 \sigma_2[n] \tag{6}$$

where $\sigma_2[n]$ denotes σ_2 composed with the shifted substitution $i \to i + n$ (here permutations are considered as words and $\nrightarrow$ is nothing else but shifted concatenation).

Example 2.1 Let $\sigma_1 = 1320 \in \mathfrak{S}_4$ and $\sigma_2 = 534120 \in \mathfrak{S}_6$. Here, we denote a permutation of $\mathfrak{S}_n$ by the word whose ith letter is the image of i under the natural action on $\{0, \ldots, n-1\}$). With this notation, we obtain

$$\sigma_1 \nrightarrow \sigma_2 = 1320978564$$

and

$$\sigma_2 \nrightarrow \sigma_1 = 5341207986$$

Clearly, it turns out that $\nrightarrow$ is not commutative.

The following proposition shows that the natural action of $\mathfrak{S}_{n+m}$ coincides with the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$.

Proposition 2.2 $(\sigma_1 \nrightarrow \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_I i.$ □

Let us introduce a similar construction for the Cartesian action: we define a map $\nsearrow\!\!\swarrow: \mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ by

$$\sigma_1 \times \sigma_2 = \prod_{i,j} c_i \times c'_j \tag{7}$$

where $\sigma_1 = c_1 \cdots c_k$ (resp. $\sigma_2 = c'_1 \cdots c'_{k'}$) is the decomposition of σ_1 (resp. σ_2) into a product of cycles and

$$c \times c' = \prod_{s=0}^{l \wedge l' - 1} (\phi(s, 0), \phi(s+1, 1) \cdots, \phi(s + l \vee l' - 1, l \vee l' - 1)), \tag{8}$$

where $\wedge$ denotes the gcd, $\vee$ denotes the lcm, $c = (i_0, \cdots, i_{l-1})$, $c' = (j_0, \cdots, j_{l'-1})$ are two cycles and $\phi(k, k') = i_{k \bmod l} + nj_{k' \bmod l'}$. Just like the Intransitive action, the Cartesian action coincides with the natural action.

Proposition 2.3 $(\sigma_1 \times \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_C i$.

Proof — From (7), it suffices to prove the property only when $\sigma_1 = c$ and $\sigma_2 = c'$ are two cycles. But as (8) is equivalent to

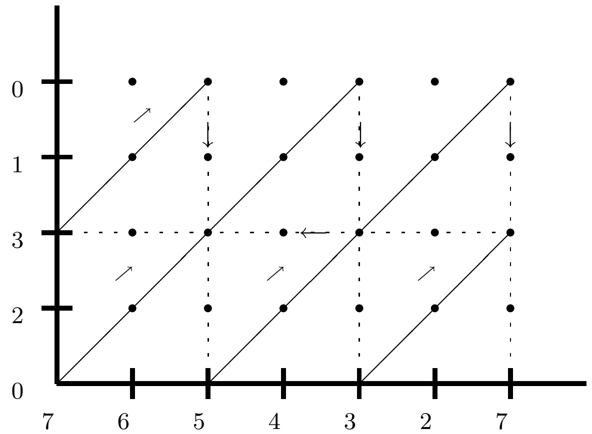
$$\begin{aligned} c \times c' = & \prod_{s=0}^{l \wedge l' - 1} (i_s + nj_0, (c, c') \circ_C (i_s + nj_0), \ldots, (c^{l \vee l' - 1}, c'^{l \vee l' - 1}) \circ_C (i_s + nj_0)) \\ = & \prod_{s=0}^{l \wedge l' - 1} (i_s + nj_0, c \circ_C i_s + nc' \circ_N j_0, \ldots, c^{l \vee l' - 1} \circ_N i_s + nc'^{l \vee l' - 1} \circ_N j_0)), \end{aligned}$$

which completes the proof. □

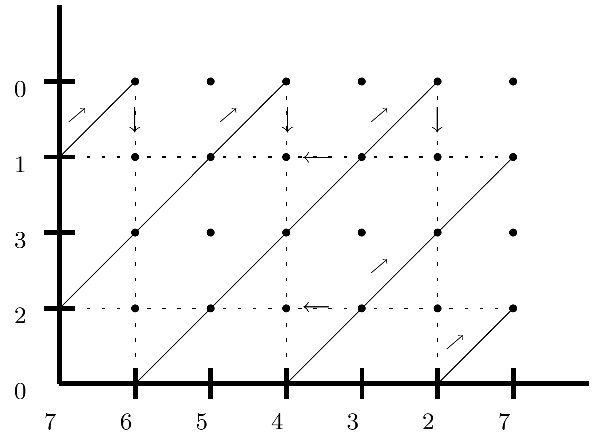
Example 2.4 Consider the two permutations $\sigma_1 = 2031 \in \mathfrak{S}_4$ and $\sigma_2 = 01723456 \in \mathfrak{S}_8$. The permutation σ_1 consists of a unique cycle $c_1 = (0, 2, 3, 1)$ and $\sigma_2 = c'_1 c'_2 c'_3$ is the product of the three cycles $c'_1 = (0)$, $c'_2 = (1)$ and $c'_3 = (7, 6, 5, 4, 3, 2)$. Hence, the permutation $\sigma_1 \times \sigma_2$ is the product of four cycles given by

1. $c_1 \times c'_1 = (0, 2, 3, 1)$
2. $c_1 \times c'_2 = (4, 6, 7, 5)$
3. $c_1 \times c'_3 = (28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)$.

To illustrate proposition 2.3, it suffices to draw the cycles in the Cartesian product $\{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ whose elements are re- labelled $(i, j) \to i + nj$. For example, the two cycles appearing in $c_1 \times c'_3$ give the following partition of $\{0, 1, 2, 3\} \times \{2, 3, 4, 5, 6, 7\}$.



(28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)



(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)

On the other hand, the permutation $\sigma_2 \times \sigma_1$ is the product of the four cycles

1. $c'_1 \times c_1 = (0, 16, 24, 8)$
2. $c'_2 \times c_1 = (1, 17, 25, 9)$
3. $c'_3 \times c_1 = (7, 22, 29, 12, 3, 18, 31, 14, 5, 20, 27, 10)(6, 21, 28, 11, 2, 23, 30, 13, 4, 19, 26, 15)$

Clearly, $\sigma_1 \times \sigma_2 \neq \sigma_2 \times \sigma_1$: the law $\times$ is not commutative.

2.3 Algebraic structure

The advantage of the new structures over the ones defined in section 2.1 consists in the omission of the operations over the groups. Hence, algebraic properties come to light quite naturally. First, the two laws are associative.

Proposition 2.5 *Associativity*
Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

1. $\sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$

2. $\sigma_1 \times (\sigma_2 \times \sigma_3) = (\sigma_1 \times \sigma_2) \times \sigma_3$

Proof — 1) Set $\eta_1 = \sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3)$ and $\eta_2 = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$. One has

$$\eta_1 \circ_N i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \leq i \leq n-1 \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \leq i \leq m+n-1 \\ \sigma_3 \circ_N (i-n-m) + n + m & \text{if } n+m \leq i \leq n+m+p-1 \end{cases}$$

for each $0 \leq i \leq n+m-1$, and the same holds for $\eta_2 \circ_N i$. It follows that $\eta_1 = \eta_2$.
2) The strategy is the same. First, we set $\eta_1 = \sigma_1 \times (\sigma_2 \times \sigma_3)$ and $\eta_2 = (\sigma_1 \times \sigma_2) \times \sigma_3$. The action of η_1 can be computed as follows

$$\eta_1 \circ_N (i + ni') = \sigma_1 \circ_N i + n(\sigma_2 \times \sigma_3) \circ_N i' = \sigma_1 \circ_N i + n\sigma_2 \circ_N j + nm\sigma_3 \circ_N k$$

where $0 \leq i \leq n-1$, $0 \leq i' \leq mp-1$, $0 \leq j \leq m-1$ and $0 \leq k \leq p-1$.
On the other hand, the action of η_2 is

$$\eta_2 \circ_N (k' + nmk) = (\sigma_1 \times \sigma_2) \circ_N k' + nm\sigma_3 \circ_N k = \sigma_1 \circ_N i + n\sigma_2 \circ_N j + nm\sigma_3 \circ_N k$$

where $0 \leq i \leq n-1$, $0 \leq j \leq m-1$, $0 \leq k \leq p-1$ and $0 \leq k' \leq nm-1$. Hence, $\eta_1 \circ_N i = \eta_2 \circ_N i$ for $0 \leq i \leq nmp-1$ and $\eta_1 = \eta_2$. $\square$

From example 2.1 and 2.4, neither $\nrightarrow$ nor $\times$ is commutative. But, one has the property of left distributivity.

Proposition 2.6 *Semi-distributivity*
Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be three permutations

$$\sigma_1 \mathbin{\times\!\!\!\!\searrow} (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \nrightarrow (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3)$$

Proof — We use the same method as in the proof of proposition 2.5. First, let us define $\eta_1 = \sigma_1 \mathbin{\times\!\!\!\!\searrow} (\sigma_2 \nrightarrow \sigma_2)$ and $\eta_2 = (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \nrightarrow (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3)$. The action of η_1 is

$$\eta_1 \circ_N (i+nj) = \eta_1 \circ_N i + n(\sigma_2 \nrightarrow \sigma_3) \circ_N j = \begin{cases} \sigma_1 \circ_N i + n\sigma_2 \circ_N j & \text{if } 0 \leq j \leq m-1 \\ \sigma_1 \circ_N i + n\sigma_3 \circ_N (j-m) + m & \text{if } m \leq j \leq p+m-1 \end{cases} \tag{9}$$

where $0 \leq i \leq n-1$ and $0 \leq j \leq m+p-1$.
On the other hand, one has

$$\eta_2 \circ_N k = \begin{cases} (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \circ_N k & \text{if } 0 \leq k \leq nm-1 \\ (\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3) \circ_N (k-nm) + nm & \text{if } nm \leq k \leq n(m+p)-1 \end{cases}. \tag{10}$$

If $0 \leq k \leq mn-1$, we set $k = i + nj$ where $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$. Hence,

$$(\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_2) \circ_N k = \sigma_1 \circ_N i + n\sigma_2 \circ_N j. \tag{11}$$

Similarly, if $nm \leq k \leq n(m+p)-1$, we set $(k-nm) = i+nj$ where $0 \leq i \leq n-1$ and $0 \leq j \leq p-1$. Hence,

$$(\sigma_1 \mathbin{\times\!\!\!\!\searrow} \sigma_3) \circ_N (k-nm) + nm = \sigma_1 \circ_N i + n(\sigma_3 \circ_N (j-m) + m). \tag{12}$$

Substituting (11) and (12) in (10), one recovers the right hand side of (9). It follows immediately that $\eta_1 = \eta_2$. $\square$

The two laws can be extended by linearity to the graded vector space $\bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n]$ and endow this space with a structure of 2-associative algebra. In the next section, we construct a product $\star$ in *Sym* (the algebra of symmetric functions) defined on the power sums and appearing when one examines the cycle index polynomial of a Cartesian product. This product is the image of $\mathbin{\times\!\!\!\!\searrow}$ under a particular morphism. We will prove that this last property implies the associativity and the distributivity of $\star$ over $\times$ (the natural product in *Sym*) and $+$.

3 Cycle index algebra

3.1 Cartesian product in *Sym*

We first construct a 2-associative morphism $\bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n] \mapsto Sym$ (a 2-associative algebra is just a vector space equipped with 2 associative laws [9]).
The arrow maps a permutation $\sigma \in \mathfrak{S}_n$ to a product of power sums. For $j \geq 1$, let $c_j(\sigma)$ be the number of cycles in σ of length j and set

$$\mathfrak{Z}(\sigma) = \prod_{j=0}^{\infty} \psi_j^{c_j(\sigma)} \tag{13}$$

where ψ_i denotes the ith power sum symmetric function. We claim that $\mathfrak{Z}$ is a morphism mapping $\nrightarrow$ to $\times$ (the usual product in *Sym*) and that $\mathbin{\times\!\!\!\!\searrow}$ is compatible with $\mathfrak{Z}$ to the extent that there exists an associative law on *Sym* such that $\mathfrak{Z}$ is also a morphism mapping it to $\mathbin{\times\!\!\!\!\searrow}$. This second law is given on the power sums basis by

$$\prod_{1\leq i\leq\infty} \psi_i^{\alpha_i} \star \prod_{1\leq j\leq\infty} \psi_j^{\beta_j} = \prod_{1\leq i,j\leq\infty} \psi_{i\vee j}^{\alpha_i\beta_j(i\wedge j)} \tag{14}$$

(the sequences $(\alpha_i)_{i\geq 1}$, $(\beta_j)_{j\geq 1}$ have finite support). It is straightforward to check that

Proposition 3.1 *i) The mapping* $\mathfrak{Z} : \bigoplus_{n\geq 0}\mathbb{Q}[\mathfrak{S}_n] \mapsto Sym$ *is a morphism of 2-associative algebras sending the two laws* $\dotplus$; $\boxtimes$ *respectively to* $\times$; $\star$ *(recall that* $\times$ *denotes the usual product of Sym).*
More precisely, for $\sigma,\ \tau \in \sqcup_{n\geq 0}\mathfrak{S}_n = \mathfrak{S}$ *one has*

$$\mathfrak{Z}(\sigma \dotplus \tau) = \mathfrak{Z}(\sigma)\mathfrak{Z}(\tau)\ ;\ \mathfrak{Z}(\sigma \boxtimes \tau) = \mathfrak{Z}(\sigma) \star \mathfrak{Z}(\tau) \tag{15}$$

ii) The law $\star$ *is associative, commutative and distributive over* $\times$.

Proof — i) For the first relation of (15), one just notices that $c_j(\sigma \dotplus \tau) = c_j(\sigma) + c_j(\tau)$. For the second relation, one observes that the Cartesian product of a i-cycle and a j-cycle produces $i \wedge j$ cycles of length $i \vee j$. Thus, one has $c_r(\sigma \boxtimes \tau) = \sum_{p\vee q=r}(p\wedge q)c_p(\sigma)c_q(\tau)$, whence (15).
ii) When $\sigma \in \mathfrak{S}_n$ is a cycle of maximum length, one has $\mathfrak{Z}(\sigma) = \psi_n$, hence the image of $\mathfrak{Z}$ contains also all the products of power sums and we get $Im(\mathfrak{Z}) = Sym$. Then, by proposition 3.1(i), $\star$ is distributive on the left over $\times$. Complete distributivity follows from commutativity of $\star$, which straightforwardly follows from the definition. $\square$

The following structural result goes into particulars of the distributivity of $\star$ over $\times$.

Proposition 3.2 *Let P be the set of products of power sums (i.e.* $P = \{\prod_{i=1}^{\infty}\psi_i^{\alpha_i}\}_{(\alpha_i)_{i\geq 1}\in\mathbb{N}^{\mathbb{N}^*}}$*). Then P is closed by* $\times$ *and* $\star$ *and more precisely* $(P,\times,\star)$ *is isomorphic to a subsemiring of the* $\mathbb{Z}$*-algebra* $\mathbb{Z}[\mathbb{N}^{\mathfrak{p}}]$ *of the monoid* $(\mathbb{N}^{\mathfrak{p}}, sup)$ *(where* $\mathfrak{p}$ *stands for the set of prime numbers).*

Proof — The fact that P is closed by $\times$ and $\star$ follows from the definition and (14). Now P contains the two units (1 and ψ_1), therefore (as a consequence of the properties established for the laws $\times$, $\star$) it is a semiring. For every $p \in \mathfrak{p}$ and $n \in \mathbb{N}^*$, let $\nu_p(n)$ be the exponent of p in the decomposition of n in prime factors ($n = \prod_{p\in\mathfrak{p}} p^{\nu_p(n)}$). Define an arrow $\phi : P \to \mathbb{Z}[(\mathbb{N}^{\mathfrak{p}}]$ by

$$\phi(\prod_{1\leq i\leq\infty}\psi_i^{\alpha_i}) = \sum_{1\leq i\leq\infty} i\alpha_i(p \mapsto \nu_p(i)). \tag{16}$$

As $\phi(m_1m_2) = \phi(m_1) + \phi(m_2)$ by construction (16), it suffices to prove that
$\phi(\psi_i \star \psi_j) = \phi(\psi_i) \times_s \phi(\psi_j)$ where $\times_s$ stands for the product in $\mathbb{Z}[(\mathbb{N}^{(\mathfrak{p})}, sup)]$. But

$$\phi(\psi_i \star \psi_j) = \phi(\psi_{i\vee j}^{i\wedge j}) = (i\wedge j)\phi(\psi_{i\vee j}) = (i\wedge j)(i\vee j)(p \mapsto \nu_p(i\vee j)) =$$
$$(i\wedge j)(i\vee j)(p \mapsto sup(\nu_p(i),\nu_p(j))) = ij(p \mapsto sup(\nu_p(i),\nu_p(j))) = \phi(\psi_i) \times_s \phi(\psi_j).$$

The arrow being clearly into the claim is proved. $\square$

3.2 Cycle index polynomial

Let $\mathfrak{S} = \bigsqcup_{n\geq 0}\mathfrak{S}_n$ be the disjoint union of all the symmetric groups and $\mathfrak{S}_{sg} = \bigcup_{n\geq 0}(\mathfrak{S}_n)_{sg}$ be the set of all the subgroups of all symmetric groups (*i.e.* the set of all permutation groups over some interval $[1..n]$). For simplicity, we identify a permutation group $G \in (\mathfrak{S}_n)_{sg}$ with its action $(G, \{0,\ldots,n-1\})$ (see section 2.1). Laws $\dotplus$ and $\boxtimes$ can be defined over $\mathfrak{S}_{sg}$ by

$$G_1 \dotplus G_2 := (G_1 \times G_2, \{0,\ldots,n+m-1\}) \tag{17}$$

where G_1 acts on $\{0,\ldots,n-1\}$ and G_2 acts on $\{n,\ldots,n+m-1\}$ and

$$G_1 \boxtimes G_2 := (G_1 \times G_2, \{0,\ldots,nm-1\}) \tag{18}$$

where the action on $\{0,\ldots,nm-1\}$ is given by $(\sigma_1,\sigma_2)k = \phi^{-1}((\sigma_1,\sigma_2)\phi(k))$, the map ϕ being the bijection $\phi : \{0,\ldots,nm-1\} \to \{0,\ldots,n-1\}\times\{0,\ldots,m-1\}$ defined by $\phi(i+nj) = (i,j)$ if $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$ and $(\sigma_1,\sigma_2)(i,j) = (\sigma_1 i, \sigma_2 j)$. Note that both $\dotplus$ and $\boxtimes$ are

associative but $\times$ is not distributive over $\nrightarrow$.
Let $Z : \mathfrak{G}_{sg} \to Sym$ be defined by

$$Z(G) = \mathfrak{Z}\left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\right). \tag{19}$$

Polyà's cycle index polynomial of G is defined to be $Z(G)$.

Example 3.3 1. The cycle index of the symmetric group $\mathfrak{S}_n$ is $Z(\mathfrak{S}_n) = h_n$.

2. The cycle index of the alternating group A_n is $Z(A_n) = h_n + e_n$.

Here h_n (resp. e_n) denotes a complete (resp. elementary) symmetric function. These examples appear as exercices in [10] (ex.9 p 29).

Since $\mathfrak{Z}$ is a morphism of 2-associative algebra, one recovers the classical relations (see [3])

$$Z(G_1 \nrightarrow G_2) = Z(G_1)Z(G_2) \tag{20}$$
$$Z(G_1 \times G_2) = Z(G_1) \star Z(G_2) \tag{21}$$

Example 3.4 1. The cycle index polynomial of the Intransitive product of two symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_m$ is

$$Z(\mathfrak{S}_n \nrightarrow \mathfrak{S}_m) = h_n h_m.$$

2. The cycle index polynomial of the Cartesian product of two symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_m$ is

$$Z(\mathfrak{S}_n \times \mathfrak{S}_m) = h_n \star h_m = \sum_{\substack{|\lambda|=n,\\ |\rho|=m}} m_\lambda \star m_\rho = \sum_{\substack{|\lambda|=n,\\ |\rho|=m}} \frac{1}{z_\lambda z_\rho}\prod_{i,j}\psi_{\lambda_i\vee\rho_j}^{\lambda_i\wedge\rho_j},$$

where m_λ denotes a monomial symmetric function and $z_\lambda = \prod i^{n_i} n_i!$ if n_i is the number of parts of λ equal to i.

3.3 Enumeration of a type of Feynman diagrams related to the Quantum Field Theory of partitions

The cycle index polynomials are classic tools used in combination with Polyà's theorem, for the enumeration of unlabelled objects. Let us recall the general process. Consider a permutation group G acting on a finite set $X = \{x_1, \cdots, x_n\}$. Let $L = \{l_0, \dots, l_p, \dots\}$ (possibly infinite) be another set, and $f : X \to L$. The *type* $t(f)$ of f is the vector $(i_0, \dots, i_p, \dots)$ where i_k is the number of elements of X whose image by f is l_k. The *shape* $s(f)$ of f is the partition obtained by sorting in the decreasing order $t(f)$ and erasing the zeroes. For example, a function f having the type $t(f) = (0, 1, 0, 9, 1, 2, 0, \dots, 0, \dots)$ has the shape $s(f) = (9, 2, 1, 1)$. The number $d_\lambda^s(G, L)$ of G-classes on L^X with the shape λ is the coefficient of m_λ in the expansion of $Z(G)$ in the basis of monomial symmetric functions:

$$Z(G) = \sum_\lambda d_\lambda^s(G, L) m_\lambda. \tag{22}$$

Now, let us apply this method to enumerate the Feynman diagrams arising in the expansion of formula (1). These diagrams are bipartite finite graphs with no isolated vertex, and edges weighted with integers. First, we enumerate all bipartite finite graphs with edges weighted with integers. Let n and m be the numbers of vertices in each of the two parts. We consider the edges as a function e from $\{0, \dots, n-1\} \times \{0, \dots, m-1\}$ to $\mathbb{N}$. The type (resp. the shape) of a graph is the type (resp. the shape) of its edges, *i.e.* $t(e)$ (resp. $s(e)$). The number $d_\lambda(n, m)$ of graphs with type λ is equal to the number of orbits with type λ, for the action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\mathbb{N}^{\{0,\dots,n-1\}\times\{0,\dots,m-1\}}$. Hence, the generating function of the shape is

$$g(n, m) := \sum_\lambda d_\lambda^s(n, m) m_\lambda = Z(\mathfrak{S}_n) \star Z(\mathfrak{S}_m) \tag{23}$$

Specializing the symmetric functions appearing in (23) to the alphabet $\{y_0,\dots,y_k,\dots,\}$, the coefficient $d_I^t(n,m)$ of $\prod y_k^{i_k}$ in the expansion of $g(n,m)$ is equal to the number of graphs with type $I=(i_0,\dots,i_k,\dots)$,

$$g(n,m)=\sum_{I=(i_0,\dots,i_p,\dots)} d_I^t(n,m)\prod_{k=0}^{\infty} y_k^{i_k}. \tag{24}$$

Note that one can enumerate graphs having edges weighted with integers less than or equal to p by specializing to the finite alphabet $\{y_0,\dots,y_p\}$.
Let us define the generating series of the type of our Feynman diagrams

$$F(n,m):=\sum_{I=(i_0,\dots,i_p,\dots)} f_I^t(n,m)\prod_{k=0}^{\infty} y_k^{i_k}, \tag{25}$$

where $f_I^t(n,m)$ denotes the number of Feynman diagrams of type I. Remark that $F(n,m)$ is a symmetric function over the alphabet $\{y_1,\dots,y_p,\dots\}$ but not over $\{y_0,\dots,y_p,\dots\}$.

Example 3.5 Let us give the first examples of generating series for weight in $\{0,1,2\}$.

1. $F(1,1)=y_1+y_2$
2. $F(2,1)=F(1,2)=y_1^2+y_1y_2+y_2^2$
3. $F(2,2)=y_0^2y_1^2+y_0^2y_2^2+y_0^2y_1y_2+y_0y_1^3+3y_0y_1^2y_2+3y_0y_1y_2^2+y_0y_2^3+y_1^4+y_1^3y_2+3y_1^2y_2^2+y_1y_2^3+y_2^4$

One can remark that under this specialization,

$$F(2,2)+F(2,1)y_0^2+F(1,2)y_0^2+F(1,1)y_0^3+y_0^4=3m_{22}+m_4+3m_{211}+m_{31}=g(2,2).$$

The latter equality could be stated in a more general setting.

Theorem 3.6 *One has the following decomposition of the cycle index polynomial.*

$$Z(\mathfrak{S}_n \times\!\!\!\!\searrow \mathfrak{S}_m)=y_0^{nm}+\sum_{(1,1)\le_{lex}(k,p)\le_{lex}(n,m)} F(k,p)y_0^{nm-kp}. \tag{26}$$

Proof — It suffices to remark that a bipartite graph is either a graph without isolated vertex or the union of some isolated vertex and a smaller bipartite graph. $\square$

This yields a nice induction formula for the $F(n,m)$'s.

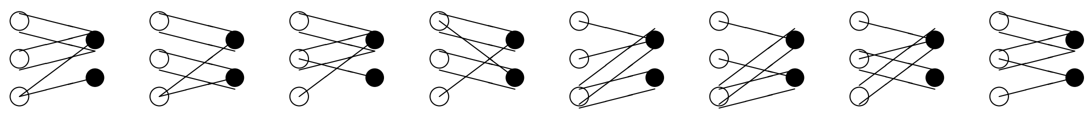
Example 3.7 From theorem 3.6, one has

$$F(3,2)=Z(\mathfrak{S}_3 \times\!\!\!\!\searrow \mathfrak{S}_2)-F(3,1)y_0^3-F(2,2)y_0^2-F(2,1)y_0^4-F(1,2)y_0^4-F(1,1)y_0^5-y_0^6.$$

From example 3.5, it suffices to compute $F(3,1)=y_1^3+y_2^3$ to enumerate Feynman diagrams whose edges are weighted by 0, 1 or 2. After simplification, one obtains

$$\begin{aligned}F(3,2) \;=\;& y_2^6+y_2^5y_1+3y_2^4y_1+3y_2^4y_1y_0+2y_2^4y_0^2+3y_2^3y_1^3+6y_2^3y_1^2y_0+5y_2^3y_1y_0^2\\ &+y_2^3y_0^3+3y_2^2y_1^4+3y_2^2y_1^3y_0+8y_2^2y_1^2y_0^2+3y_2^2y_1y_0^3+y_2y_1^5+3y_2y_1^4y_0+5y_2y_1^3y_0^2\\ &+3y_2y_1^2y_0^3+y_1^6+y_1^5y_0+y_1^3y_0^3+2y_1^2y_0^4.\end{aligned}$$

For example, there are 8 $(2,2,2)$- Feynman diagrams:



4 Non commutative realizations

4.1 Free quasi-symmetric cycle index algebra

Let $(A,<)$ be an ordered alphabet and $w \in A^*$ a word of length n. One denotes by $Std(w)$, the permutation $\sigma \in \mathfrak{S}_n$ defined by

$$\sigma(i) = (\text{Number of letters} = w[i] \text{ in } w[1..i] + \text{number of letters} < w[i] \text{ in } w) \tag{27}$$

Recall that the algebra **FQSym** is defined by one of its bases, indexed by $\mathfrak{S}$ and defined as follows

$$\mathbf{F}_\sigma = \sum_{Std(w)=\sigma^{-1}} w \in \mathbb{Z}\langle\langle A \rangle\rangle \tag{28}$$

In [5], it is shown that **FQSym** is freely generated by the $\mathbf{F}_\sigma$ where σ runs over the connected permutations (see [4]) (*i.e.* permutations such that $\sigma([1,k]) \neq [1,k]$ for each k). The algebra **FQSym** is spanned by a linear basis, $\{\mathbf{F}^\sigma\}_{\sigma\in\mathfrak{S}}$, whose product implements the Intransitive action $\nrightarrow$:

$$\mathbf{F}^\sigma = \mathbf{F}_{\sigma_1} \cdots \mathbf{F}_{\sigma_n} \tag{29}$$

where $\sigma = \sigma_1 \nrightarrow \cdots \nrightarrow \sigma_n$ is the maximal factorisation of σ in connected permutations. As a consequence of this definition, one has

$$\mathbf{F}^\sigma \mathbf{F}^\tau = \mathbf{F}^{\sigma \nrightarrow \tau}. \tag{30}$$

This naturally induces an isomorphism of algebras

$$\begin{array}{rcl} \underline{\mathfrak{Z}} : \left(\bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n], \nrightarrow, + \right) & \to & (\mathbf{FQSym}, ., +) \\ \sigma & \mapsto & \mathbf{F}^\sigma. \end{array} \tag{31}$$

One defines the product $\star$ on **FQSym** by $\mathbf{F}^\sigma \star \mathbf{F}^\tau := \mathbf{F}^{\sigma \times \tau}$. By this way, $\underline{\mathfrak{Z}}$ becomes a morphism of 2-associative algebras. Furthermore, $\star$ is associative, distributive over the sum and semi-distributive over the shifted concatenation.

4.2 Free quasi-symmetric Polyà cycle index polynomial

Let G be a permutation group. The *free quasi-symmetric Polyà cycle index polynomial* of G is its image by $\underline{Z} : \mathfrak{S}_{sg} \to \mathbf{FQSym}$ defined by

$$\underline{Z}(G) := \underline{\mathfrak{Z}}\left(\frac{1}{|G|} \sum_{\sigma\in G} \sigma \right) \mathbf{F}^\sigma. \tag{32}$$

Note 4.1 *There is another basis of* **FQSym** *indexed by permutations, namely* $\{\mathbf{G}^\sigma\}_{\sigma\in\mathfrak{S}}$. *It is obtained by setting* $\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}}$ *and applying the same construction as above (30) to get a basis multiplicative with respect to* $\nrightarrow$*, then*

$$\mathbf{G}^\sigma = \mathbf{G}_{\sigma_1} \cdots \mathbf{G}_{\sigma_n} \tag{33}$$

where $\sigma = \sigma_1 \nrightarrow \cdots \nrightarrow \sigma_n$ *is the maximal factorisation of* σ *into connected permutations. In this case,* σ^{-1} *splits maximally into* $\sigma_1^{-1} \nrightarrow \cdots \nrightarrow \sigma_n^{-1}$*, so one has also* $\mathbf{G}^\sigma = \mathbf{F}^{\sigma^{-1}}$ *and formula (34) can be rewritten*

$$\underline{Z}(G) := \underline{\mathfrak{Z}}\left(\frac{1}{|G|} \sum_{\sigma\in G} \sigma \right) \mathbf{G}^\sigma. \tag{34}$$

The polynomial $\underline{Z}(G)$ has properties similar to that of $Z(G)$, in particular regarding the laws $\nrightarrow$ and $\times$.

Proposition 4.2 *Let $G_1, G_2 \in \mathfrak{S}_{sg}$ be two permutation groups, one has*

1. $\underline{Z}(G_1 \nrightarrow G_2) = \underline{Z}(G_1)\underline{Z}(G_2)$.
2. $\underline{Z}(G_1 \times G_2) = \underline{Z}(G_1) \star \underline{Z}(G_2)$.

Consider the morphism, $z : \mathbf{FQSym} \to Sym$ defined by $z(\mathbf{F}^\sigma) = \mathfrak{Z}(\sigma)$. Note that it is not a morphism of Hopf algebra because $z(\mathbf{F}^{231}) = \psi_3$.
The following diagram is commutative

$$\begin{array}{ccc} \mathfrak{S}_{sg} & \xrightarrow{\underline{Z}} & \mathbf{FQSym} \\ Z\downarrow & {}^{z}\swarrow & \uparrow \underline{\mathfrak{Z}} \\ Sym & \underset{\mathfrak{Z}}{\longleftarrow} & \bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n] \end{array} \tag{35}$$

Example 4.3 1. The free quasi-symmetric cycle index of $\mathfrak{S}_n$ is

$$\mathbf{H}_n := \underline{Z}(\mathfrak{S}_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \mathbf{F}^\sigma.$$

One can consider it as a free quasi-symmetric analogue of the complete symmetric function h_n: indeed $z(\mathbf{H}_n) = Z(\mathfrak{S}_n) = h_n$.

2. One can define free quasi-symmetric analogues of elementary symmetric functions considering the cycle index polynomial of the alternative groups:

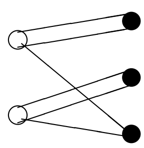
$$\mathbf{E}_n := \underline{Z}(A_n) - \underline{Z}(\mathfrak{S}_n).$$

We get $z(\mathbf{E}_n) = Z(A_n) - Z(\mathfrak{S}_n) = e_n$.

3. The knowledge of analogues of other symmetric functions should be useful to understand the combinatorics of free quasi-symmetric cycle index. In particular, it should be interesting to find free quasi-symmetric functions whose images by z are the monomial symmetric functions.

4.3 Realizations in MQSym

We will call *labelled diagrams* the Feynman diagrams as above but with p white (resp. q black) spots labelled bijectively by $[1..p]$ (resp. by $[1..q]$). When one draws such a diagram, one implicitly assumes that the labelling goes from top to bottom.



Labelled diagram of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

Now, to such a $p \times q$ *labelled diagram* we can associate a matrix in $\mathbb{N}^{p\times q}$ and this correspondence is one-to-one. The condition that no vertex be isolated is equivalent to the condition that there be no complete line or column of zeroes, *i.e.* the representative matrix is *packed* [5]. In the same way, the diagrams are in one-to-one correspondence with the classes of packed matrices under the permutations of lines and columns as shown below (the vertical arrows are then one-to-one)

$$\begin{array}{ccc} \text{Packed matrices} & \xrightarrow{Class} & \text{Classes of packed matrices} \\ \downarrow & & \downarrow \\ \text{Labelled diagrams} & \longrightarrow & \text{Diagrams} \end{array} \tag{36}$$

There is an interesting structure of Hopf algebra (in fact an envelopping algebra) over the diagrams [6] which can be pulled back in a natural way to labelled diagrams.
The correspondence described above allows to construct a new Hopf algebra structure on **MQSym** and a Hopf algebra structure on the space spanned by the classes.

5 Conclusion

Other realizations in Hopf algebras seem feasible. For example, let us consider the Hopf algebras of graphs $GQSym^{110}$ and $GTSym^{110}$ defined in [11]. An interesting mapping from $\bigoplus_{n\geq 0}\mathbb{Q}[\mathfrak{S}_N]$ to $GQSym^{110}$ or $GTSym^{110}$ can be constructed sending each cycle to an equivalent loop.

References

[1] C.M. Bender, D.C. Brody and B.K. Meister, *Quantum field theory of partitions*, *J.Math. Phys.* **40** (2000), 3239.

[2] C.M. Bender, D.C. Brody, and B.K. Meister, Combinatorics and field theory, *Twistor Newsletter* **45** (2000), 36.

[3] P.J. Cameron, D.A. Gewurz, F. Merola, *Product action*, Preprint, `http://www.maths.qmw.ac.uk/~pjc/preprints/product.pdf`

[4] L. Comtet, *Sur les cœfficients de l'inverse de la série formelle* $\sum n!t^n$, C.R. Acad. Sci. Paris, Ser. A **275** (1972), 569-572.

[5] G. Duchamp , F. Hivert , J.-Y. Thibon : *Non commutative functions VI: Free quasi-symmetric functions and related algebras*, International Journal of Algebra and Computation **12** (2000), N 5.

[6] G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak, *Hopf algebras and Lie group structures inherent in combinatorial field theories*, New symmetries in Mathematical Physics (CIRM - November 22 - 26th 2004). Organizers: R. Kerner (Prof., Univ. Paris 6), J. Kijowski (Prof., Univ. Varsovie)

[7] G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak, *One-parameter groups and combinatorial physics*, Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), Porto-Novo (Benin), November 2003. arXiv : `quant-ph/0401126`

[8] K.A. Penson, P. Blasiak, G. Duchamp, A. Horzela and A.I. Solomon, *Hierarchical Dobiński-type relations via substitution and the moment problem*, J. Phys. A, to appear (2004), arXiv : `quant-ph/0312202`

[9] J.-L. Loday and M. Ronco, *On the structure of cofree Hopf algebras*, preprint arXiv :`math.QA/0405330`

[10] I.G. Macdonald, Symmetric functions and Hall polynomial, second edition, Clarendon Press, Oxford, 1995.

[11] J.-C. Novelli and J.-Y. Thibon, *Algèbres de Hopf de graphes*, C.R. Acad. Sci. Paris Ser. I, to appear Preprint: `http://www-igm.univ-mlv.fr/ jyt/ARTICLES/graphs.ps`.

[12] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen.*, Acta Math. **68** (1937), 145-254.

[13] A.I. Solomon, P. Blasiak, G. Duchamp, A. Horzela and K.A. Penson, *Combinatorial Physics, Normal Order and Model Feynman Graphs*, Proceedings of the Symposium 'Symmetries in Science XIII', Bregenz, Austria, 2003. Kluwer Publ. (2004) in press. arXiv : `quant-ph/0310174`